MOMENTLESS ELASTIC REINFORCED SHELLS WITH A ZERO GAUSSIAN CURVATURE

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In [1], a number of statements were formulated of the problem of the attainment of a momentless stressed state in elastic reinforced shells with an arbitrary form of their middle surface. The present work is devoted to the solution of three of the statements advanced in [1], for the case where the middle surface has a zero Gaussian curvature.

§1. We take the calculating model of a reinforced shell advanced in [1]. We refer the middle surface of the shell to the curvilinear orthogonal coordinates α and β, coinciding with the lines of curvature; here, we denote by A and B the coefficients of the first quadratic form of the surface, and by R_1 and R_2 the principal radii of curvature.

Let the middle surface have a zero Gaussian curvature ($R_1=\infty$). If the shell works under a momentless stressed state, the following must be satisfied:

The equations of equilibrium

$$(BT_1)_{,\alpha}-B_{,\alpha}T_2+T_{3,\beta}=-Bp_1;\ T_{2,\beta}+(BT_3)_{,\alpha}+B_{,\alpha}T_3=-Bp_2,\quad T_2=Rp_3; \tag{1.1}$$

the relationships of elasticity

$$\begin{gathered}\mathbf{T}=2h\|a_{km}\|\boldsymbol{\varepsilon},\ \boldsymbol{\varepsilon}=0.5h^{-1}\|b_{km}\|\mathbf{T};\\ \mathbf{T}=\|T_1T_2T_3\|',\quad \boldsymbol{\varepsilon}=\|\varepsilon_1\varepsilon_2\varepsilon_3\|',\quad \|b_{km}\|=\|a_{km}\|^{-1};\end{gathered} \tag{1.2}$$

the geometric equations

$$\begin{gathered}\varepsilon_1=u_{,\alpha},\quad \varepsilon_2=B^{-1}v_{,\beta}+B^{-1}B_{,\alpha}u+wR^{-1},\ \varepsilon_3=B^{-1}u_{,\beta}+B\left(vB^{-1}\right)_{,\alpha};\\ \varkappa_1=-w_{,\alpha\alpha}=0,\quad \varkappa_2=-B^{-1}\left(B^{-1}w_{,\beta}-vR^{-1}\right)_{,\beta}-B^{-1}B_{,\alpha}u_{,\alpha}=0;\\ \tau=-B^{-1}\left(w_{,\alpha\beta}-B^{-1}B_{,\alpha}w_{,\beta}\right)+BR^{-1}\left(vB^{-1}\right)_{,\alpha}=0;\end{gathered} \tag{1.3}$$

the equations of continuity of the strains

$$(B\varepsilon_3)_{,\alpha}-\varepsilon_{1,\beta}=0,\quad [(B\varepsilon_2)_{,\alpha}-B_{,\alpha}\varepsilon_1]_{,\alpha}=0. \tag{1.4}$$

Here $R=R_2$; T_1, T_2, T_3 are the normal and shear stresses; ε_1, ε_2, ε_3 are the relative elongations along the α- and β-lines and the shear; $\varkappa_1$, $\varkappa_2$, τ are the changes in the curvature and the torsion; u, v, w are the components of the displacements along the α- and β-lines and the bending of the middle surface; p_1, p_2, p_3 are the components of the external surface load; 2h is the thickness of the shell; the length of an arc of the α-line is taken as the parameter α, from which it follows that A = 1. Here and in what follows, the index after the comma denotes the partial derivative with respect to the corresponding coordinate.

The coefficients a_{km} (k, m = 1, 2, 3) have the form [1, 2]

$$a_{ii}=\frac{a\mathrm{E}}{1-\nu^2}+\sum_{n=1}^{N}\omega_n\mathrm{E}_n l_{in}^4,\quad a_{12}=\frac{a\mathrm{E}\nu}{1-\nu^2}+\sum_{n=1}^{N}\omega_n\mathrm{E}_n l_{1n}^2 l_{2n}^2, \tag{1.5}$$

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$$a_{i3} = \sum_{n=1}^{N} \omega_n E_n l_{1n}^3 l_{jn}, \quad a_{33} = \frac{aE}{2(1-\nu)} + \sum_{n=1}^{N} \omega_n E_n l_{1n}^2 l_{2n}^2,$$
$$l_{1n} = \cos\psi_n, \quad l_{2n} = \sin\psi_n, \quad i, j = 1, 2, \quad i \neq j,$$

where n is the number of families of filaments (n = 1, 2, . . . , N); ω_n is the relative volumetric content of the filaments of this family; E_n is their Young's modulus; ψ_n is the angle between the direction of the filaments of the family and the α-line; E, ν are the Young modulus and the Poisson coefficient of the material of the binder; a is the relative volumetric content of the binder.

If ε_1, ε_2, ε_3 satisfy the equations of continuity of the strains (1.4), Eqs. (1.3) serve to determine the components of the displacement u, v, w. We shall postulate that the boundary conditions for the shell are so formulated that there exists a solution of Eqs. (1.3) with respect to u, v, w satisfying these conditions. Therefore, in what follows, Eqs. (1.3) and the boundary conditions for u, v, w are left out of the discussion.

We shall formulate and discuss further three statements of the problem of attaining a momentless stressed state in reinforced shells of zero curvature.

§2. For a shell of zero curvature, let there be given the following: the form of the middle surface, the laws of change in the thickness and the character of the anisotropy, and the momentless boundary conditions. It is required to find the dependences between the components of the external surface load which bring about a momentless stressed state in such a shell.

We shall limit ourselves to the case of cylindrical shells, with an arbitrary form of their directrices. Then, taking as the parameter β the length of an arc s of the directrices, we obtain B = 1, R = R(s), and, consequently, from the second equation of (1.4) we find

$$\varepsilon_2 = \alpha\varphi_1(s) + \varphi_2(s), \tag{2.1}$$

where $\varphi_1(s)$, $\varphi_2(s)$ are arbitrary functions of the integration.

From relationships (1.1), (2.1) we obtain

$$T_1 = (2h\varepsilon_2 - b_{22}Rp_3 - b_{23}T_3)\, b_{12}^{-1}. \tag{2.2}$$

The first two equations of (1.1) and the first equation of (1.4), using the dependences (1.2) and the third equation from (1.1), are brought to the form

$$c_{11}T_{3,\alpha} + T_{3,s} + c_{13}T_3 = c_1, \quad T_{3,\alpha} = c_2; \tag{2.3}$$
$$c_{31}T_{3,\alpha} + c_{32}T_{3,s} + c_{33}T_3 = c_3,$$

where

$$c_{11} = -b_{23}b_{12}^{-1}, \quad c_{13} = c_{11,\alpha}, \quad c_{31} = 0.5h^{-1}(b_{33} + b_{13}c_{11}),$$
$$c_{32} = -0.5h^{-1}(b_{13} + b_{11}c_{11}), \quad c_{33} = c_{31,\alpha} + c_{32,s},$$
$$c_1 = -p_1 + (b_{22}b_{12}^{-1}Rp_3)_{,\alpha} - 2(hb_{12}^{-1}\varepsilon_2)_{,\alpha}, \quad c_2 = -p_2 - (Rp_3)_{,s},$$
$$c_3 = (b_{11}b_{12}^{-1}\varepsilon_2 + c_4p_3)_{,s} - (b_{13}b_{12}^{-1}\varepsilon_2 + c_5p_3)_{,\alpha},$$
$$c_4 = 0.5h^{-1}R(b_{12} - b_{11}b_{22}b_{12}^{-1}), \quad c_5 = 0.5h^{-1}R(b_{23} - b_{13}b_{22}b_{12}^{-1}).$$

The conditions of the integrability [3] of Eqs. (2.3) have the form

$$A_1T_3 = A_2; \quad B_1T_3 = B_2, \tag{2.4}$$

where

$$A_1 = -c_{13,\alpha}; \quad B_1 = c_{33} - c_{13}c_{32};$$

$$A_2 = p_{1,\alpha} + \gamma_{11}p_2 + \gamma_{12}p_{2,\alpha} - p_{2,s} + \gamma_{13}p_3 + \gamma_{14}p_{3,\alpha} + \gamma_{15}p_{3,s} + \gamma_{16}p_{3,\alpha\alpha} + \gamma_{17}p_{3,ss} + \gamma_{18}p_{3,\alpha s} + \gamma_1;$$
$$B_2 = \gamma_{21}p_1 + \gamma_{22}p_2 + \gamma_{23}p_3 + \gamma_{24}p_{3,\alpha} + \gamma_{25}p_{3,s} + \gamma_2;$$
$$\gamma_{11} = -c_{13} - c_{11,\alpha}, \quad \gamma_{12} = -c_{11}, \quad \gamma_{13} = -R_{,ss} + R(b_{22}b_{12}^{-1})_{,\alpha} - \gamma_{11},$$
$$\gamma_{14} = -c_{11}R_{,s} - 2R(b_{22}b_{12}^{-1})_{,\alpha}, \quad \gamma_{15} = -2R_{,s} + \gamma_{11}R,$$
$$\gamma_{16} = -Rb_{22}b_{12}^{-1}, \quad \gamma_{17} = -R, \quad \gamma_{18} = -c_{11}R, \quad \gamma_1 = 2(hb_{12}^{-1}\varepsilon_2)_{,\alpha\alpha},$$
$$\gamma_{21} = c_{32}, \quad \gamma_{22} = c_{31} - c_{11}c_{32}, \quad \gamma_{23} = \gamma_{22}R_{,s} - c_{32}R(b_{22}b_{12}^{-1})_{,\alpha} + c_{4,s} - c_{5,\alpha},$$
$$\gamma_{24} = -c_{32}Rb_{22}b_{12}^{-1} - c_5, \quad \gamma_{25} = \gamma_{22}R + c_4,$$
$$\gamma_2 = 2c_{32}(hb_{12}^{-1}\varepsilon_2)_{,\alpha} - (b_{13}b_{12}^{-1}\varepsilon_2)_{,\alpha} + (b_{11}b_{12}^{-1}\varepsilon_2)_{,s}.$$

If the cylinder works in a momentless state, then system (2.3) must admit of a solution for T_3 and, consequently, the equalities (2.4) must be satisfied. Here three cases are possible. If

$$A_1 = B_1 = 0, \tag{2.5}$$

then from (2.4) we obtain

$$A_2 = B_2 = 0. \tag{2.6}$$

In this case, the function T_3 can be determined from any of the equations of (2.3), for example, from the second.

If $A_1 = 0$, $B_1 \neq 0$, then from equality (2.4) we find

$$A_2 = 0; \tag{2.7}$$

$$T_3 = B_2 B_1^{-1}. \tag{2.8}$$

It can be verified that, in order that the expression found for T_3 actually be a solution of Eqs. (2.3), we require the satisfaction of a relationship, obtained with the substitution of the expression for T_3 from (2.8) into the second equation of (2.3),

$$(B_2 B_1^{-1})_{,\alpha} - c_2 = 0. \tag{2.9}$$

If $A_1 \neq 0$, $B_1 \neq 0$, or $B_1 = 0$, then from the first equality of (2.4) we find

$$T_3 = A_2 A_1^{-1}. \tag{2.10}$$

In order for the expression found for T_3 to be a solution of Eqs. (2.3), we require the solution of relationships, obtained with the substitution of the value for T_3 from (2.10) into the first two of Eqs. (2.3) and into the second equality of (2.4),

$$(A_2 A_1^{-1})_{,s} + c_{13} A_2 A_1^{-1} - c + c_{11} c_2 = 0; \quad (A_2 A_1^{-1})_{,\alpha} - c_2 = 0, \quad B_1 A_2 A_1^{-1} - B_2 = 0. \tag{2.11}$$

Thus, depending on the starting data of the statement under consideration, the components of the external surface load, bringing about a momentless state in the cylinder, must satisfy the relationships (2.6) or (2.7), (2.9) or (2.11).

We shall illustrate the results obtained using the following example: We consider a closed cylindrical shell with a directrix of length L (we take the origin for the coordinate α at one of the bounding ends). The shell is made of an isotropic material ($a = 1$, $G_n = \omega_n E_n (aE)^{-1} = 0$, $n = 1, \ldots, N$) and has a constant thickness ($h = \text{const}$). The component p_3 of the external surface load satisfies the equality $p_3 = p = \text{const}$ ($p > 0$). At the edges of the shell, the following conditions are given:

$$T_1|_{\alpha=0} = T_1|_{\alpha=L} = 0, \quad T_3|_{\alpha=0} = -T_3|_{\alpha=L}. \tag{2.12}$$

It is required to find the laws of change of the components p_1 and p_2 of the external load with which a momentless state will be realized in the cylinder under consideration.

Since, with the conditions formulated, equalities (2.5) are satisfied, the sought components p_1 and p_2 must satisfy relationships (2.6), which, in the present case, have the form

$$p_{1,\alpha} - p_{2,s} - R_{,ss} p = 0; \quad h^{-1}(1+\nu) p_2 + \gamma_{25,s} p + \gamma_2 = 0. \tag{2.13}$$

Using dependences (1.2), (2.1), the first equation of (1.1), and the boundary conditions (2.12), from (2.13), (2.2), and the second equation of (2.3), we find

$$p_1 = 0.5\nu(1+\nu)^{-1}(\alpha - 0.5L) R_{,ss}\, p, \quad p_2 = -0.5(2+\nu)(1+\nu)^{-1} R_{,s} p; \tag{2.14}$$

$$T_1 = 0, \quad T_3 = 0.5\nu(1+\nu)^{-1}(0.5L - \alpha) R_{,s}\, p. \tag{2.15}$$

By virtue of the third equality of (1.1) and (2.15), the condition for elastic work of the material of the shell [1] can be represented in the form

$$0.5 h^{-1} p \{R^2 + [0.5\nu(1+\nu)^{-1}(0.5L - \alpha) R_{,s}]^2\}^{1/2} < \sigma_0. \tag{2.16}$$

This inequality leads to a limitation on the value of the load p. Let us apply the solutions obtained to an actual partial case, i.e., to a cylinder of elliptical transverse cross section. Then [4] $R = a_1\gamma^2(1 - \varepsilon\cos^2\varphi)^{-1/2}$, $ds = R\,d\varphi$,

$$\frac{dR}{ds} = -\frac{3\varepsilon \sin 2\varphi}{2(1-\varepsilon\cos^2\varphi)}, \quad \frac{d^2R}{ds^2} = -\frac{3\varepsilon(\cos 2\varphi - \varepsilon\cos^2\varphi)}{a_1\sqrt{1-\varepsilon\cos^2\varphi}},$$
$$\frac{d^3R}{ds^3} = \frac{6\varepsilon}{a_1^2\gamma^4}\left[1 - \frac{3}{4}\varepsilon - \frac{1}{2}\varepsilon\left(1 - \frac{1}{2}\varepsilon\right)\cos^2\varphi\right]\sin 2\varphi, \tag{2.17}$$

where $\varepsilon = 1 - \gamma^2$; $\gamma = a_2 a_1^{-1}$; a_1 and a_2 are the large and small semiaxes of the ellipse; φ is the angle formed by a normal to the ellipse and its small axis.

We introduce the dimensionless quantities $q_1 = p_1 p^{-1}$, $q_2 = p_2 p^{-1}$, $x = \alpha L^{-1}$, $l = L a_1^{-1}$,

$$z_0 = \{(Ra_1^{-1})^2 + 3\left[0.5\nu l(1+\nu)^{-1}(0.5 - x)R_{,s}\right]^2\}^{1/2}. \tag{2.18}$$

Dependences of q_1 (solid lines) and q_2 (dashed lines) on x, calculated using formulas (2.14), (2.17), are shown in Fig. 1a (with $l = 2$) and Fig. 1b (with $l = 4$). The numbers 1-9 correspond to the values $\varphi = 0$, $\pi/8$, $\pi/4$, $3/8\pi$, $\pi/2$, $5/8\pi$, $3/4\pi$, $7/8\pi$, π, the numbers without circles correspond to the case $\gamma = 1/2$, and the numbers enclosed in circles to the case $\gamma = 1/3$. For determinacy in the calculations it was assumed that $\nu = 0.3$.

In accordance with condition (2.16), the shell will work elastically only if the inequality $2h\sigma_0(a_1 p)^{-1} > z_0$ is satisfied. The maximal values of z_0 along the cross sections φ = const, calculated using formula (2.18) for the values of l, γ, φ, under consideration are given in Table 1. We note that, in any given cross section x = const, each of the quantities z_0, q_1 has identical values for points of the middle surface which are symmetrical with respect to the ellipse of the cross section, and q_2 for points which are symmetrical with respect to the center of the ellipse. With $\gamma = 1$ (round cylinder), from (2.14), (2.15), we obtain $p_1 = p_2 = T_3 = 0$.

§3. Let us consider two further statements of the problem of attaining a momentless state; the methods of solution are very similar.

First Statement. Let there be given the following: the form of the middle surface, the law of the distribution of the external surface load, the rigidities $G_k = \omega_k E_k \cdot (aE)^{-1}$ (previously reinforced, $k = 2, \ldots, N$), the angles ψ_n ($n = 1, \ldots, N$), and the momentless boundary conditions. It is required to find the laws of change in the thickness h and the rigidity G_1 (additionally reinforced) with which a momentless state will be realized in such a shell.

Second Statement. Given: the form of the middle surface, the law of the distribution of the external surface load, the thickness, the rigidities G_k (previously reinforced, $k = 3, \ldots N$), the angles ψ_n ($n = 1, \ldots, N$),

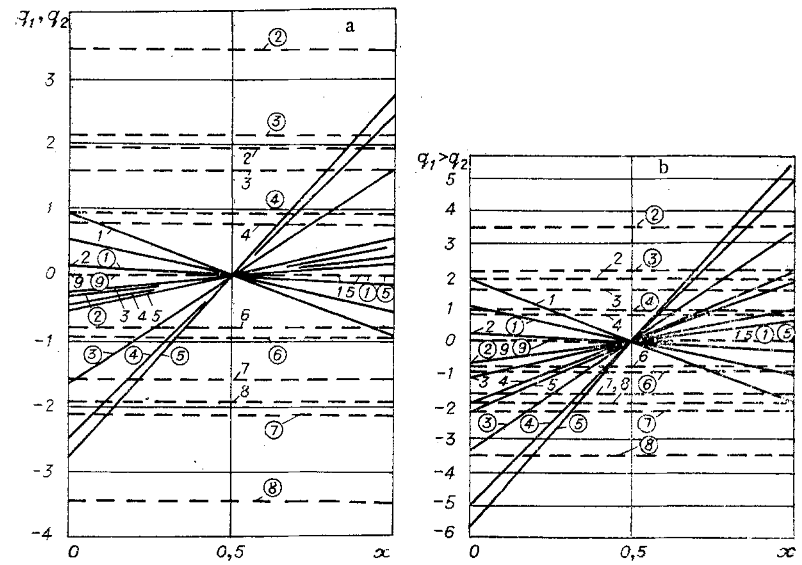

Fig. 1

TABLE 1

l	γ	$\varphi=0$	$\pi/8$	$\pi/4$	$3/8\ \pi$	$\pi/2$
2	1/2	2,0	1,240	0,621	0,347	0,250
	1/3	3,0	1,220	0,550	0,256	0,111
4	1/2	2,0	1,457	0,880	0,465	0,250
	1/3	3,0	1,821	0,996	0,454	0,111

and the momentless boundary conditions. It is required to determine the laws of change in the rigidities G_1 and G_2 of two supplementary families of filaments with which the stressed state in the shell under consideration will be momentless.

Under the conditions of both statements, the stresses T_1, T_2, T_3 are found from Eqs. (1.1), with an accuracy up to arbitrary constants determined from the boundary conditions [5].

For solution of the first statement, we represent the relationships of the elasticity (1.2), using dependences (1.5), in the form

$$\begin{aligned} a'_{11}\varepsilon_1 + G_1 e_1 l_{11}^2 - T_1(2haE)^{-1} &= -a'_{12}\varepsilon_2 - a'_{13}\varepsilon_3; \\ a'_{12}\varepsilon_1 + G_1 e_1 l_{21}^2 - T_2(2haE)^{-1} &= -a'_{22}\varepsilon_2 - a'_{23}\varepsilon_3; \\ a'_{13}\varepsilon_1 + G_1 e_1 l_{11} l_{21} - T_3(2haE)^{-1} &= -a'_{23}\varepsilon_2 - a'_{33}\varepsilon_3, \end{aligned} \tag{3.1}$$

where

$$\begin{gathered} e_1 = \varepsilon_1 l_{11}^2 + \varepsilon_2 l_{21}^2 + \varepsilon_3 l_{11} l_{21}; \\ a'_{ii} = (1-\nu^2)^{-1} + \sum_{k=2}^{N} G_k l_{ik}^4, \quad a'_{12} = \nu(1-\nu^2)^{-1} + \sum_{k=2}^{N} G_k l_{1k}^2 l_{2k}^2, \\ a'_{i3} = \sum_{k=2}^{N} G_k l_{ik}^3 l_{jk}, \quad a'_{33} = 0{,}5(1+\nu)^{-1} + \sum_{k=2}^{N} G_k l_{1k}^2 l_{2k}^2, \quad i;\ j = 1, 2, \quad i \neq j. \end{gathered} \tag{3.2}$$

Assuming $\Delta \neq 0$, from (3.1) we find

$$2h = \Delta\Delta_1^{-1}, \quad G_1 = \Delta_2(\Delta e_1)^{-1}; \tag{3.3}$$

$$\varepsilon_1 = -k_1\varepsilon_2 + k_2\varepsilon_3; \tag{3.4}$$

where

$$\begin{gathered} \Delta_i = (-1)^i\left(a'_{12}A_i + a'_{22}B_i + a'_{23}C_i\right)\varepsilon_2 + (-1)^i\left(a'_{13}A_i + a'_{23}B_i + a'_{33}C_i\right)\varepsilon_3, \\ \Delta = -(A_1T_1 + B_1T_2 + C_1T_3); \quad k_1 = \Delta^{-1}\left(a'_{12}A_3 + a'_{22}B_3 + a'_{23}C_3\right), \\ k_2 = \Delta^{-1}\left(a'_{13}A_3 + a'_{23}B_3 + a'_{33}C_3\right); \\ A_1 = \left(a'_{12}l_{11} - a'_{13}l_{21}\right)l_{21}, \quad A_2 = a'_{13}T_2 - a'_{12}T_3, \quad A_3 = (T_2l_{11} - T_3l_{21})l_{21}; \\ B_1 = \left(a'_{13}l_{11} - a'_{11}l_{21}\right)l_{11}, \quad B_2 = a'_{11}T_3 - a'_{13}T_1, \quad B_3 = (T_3l_{11} - T_1l_{21})l_{11}; \\ C_1 = a'_{11}l_{21}^2 - a'_{12}l_{11}^2, \quad C_2 = a'_{12}T_1 - a'_{11}T_2, \quad C_3 = T_1l_{21}^2 - T_2l_{11}^2. \end{gathered} \tag{3.5}$$

For solution of the second statement, we write the relationships (1.2), using dependences (1.5), in the form

$$\begin{aligned} a''_{11}\varepsilon_1 + G_1e_1l_{11}^2 + G_2e_2l_{12}^2 &= T_1(2haE)^{-1} - a''_{12}\varepsilon_2 - a''_{13}\varepsilon_3; \\ a''_{12}\varepsilon_1 + G_1e_1l_{21}^2 + G_2e_2l_{22}^2 &= T_2(2haE)^{-1} - a''_{22}\varepsilon_2 - a''_{23}\varepsilon_3; \\ a''_{13}\varepsilon_1 + G_1e_1l_{21}l_{11} + G_2e_2l_{12}l_{22} &= T_3(2haE)^{-1} - a''_{23}\varepsilon_2 - a''_{33}\varepsilon_3, \end{aligned} \tag{3.6}$$

where

$$\begin{gathered} e_i = \varepsilon_i l_{ii}^2 + \varepsilon_j l_{ji}^2 + \varepsilon_3 l_{ii} l_{ji} \quad (i, j = 1, 2; \quad i \neq j); \\ a''_{ii} = (1-\nu^2)^{-1} + \sum_{k=3}^{N} G_k l_{ik}^4, \quad a''_{12} = \nu(1-\nu^2)^{-1} + \sum_{k=3}^{N} G_k l_{1k}^2 l_{2k}^2, \\ a''_{i3} = \sum_{k=3}^{N} G_k l_{ik}^3 l_{jk}, \quad a''_{33} = 0.5(1+\nu)^{-1} + \sum_{k=3}^{N} G_k l_{1k}^2 l_{2k}^2. \end{gathered} \tag{3.7}$$

Assuming $\Delta \neq 0$, from (3.5) we find

$$G_1 = \Delta_1\Delta^{-1}, \quad G_2 = \Delta_2\Delta^{-1}; \tag{3.8}$$

$$\varepsilon_1 = F - k_1\varepsilon_2 + k_2\varepsilon_3, \tag{3.9}$$

where

$$\begin{gathered} \Delta_i = (2haE)^{-1}(A_iT_1 + B_iT_2 + C_iT_3) - \left(a''_{12}A_i + a''_{22}B_i + a''_{23}C_i\right)\varepsilon_2 - \left(a''_{13}A_i + a''_{23}B_i + a''_{33}C_i\right)\varepsilon_3, \\ \Delta = \Delta_* \sin(\psi_1 - \psi_2), \quad \Delta_* = \sum_{m=1}^{3} a''_{1m} l_m; \quad F = (2haE\Delta_*)^{-1}\sum_{m=1}^{3} T_m l_m, \end{gathered} \tag{3.10}$$

$$k_1 = \Delta_*^{-1} \sum_{m=1}^{3} a''_{m2} l_m, \quad k_2 = -\Delta_*^{-1} \sum_{m=1}^{3} a''_{m3} l_m;$$
$$l_1 = l_{21} l_{22}, \quad l_2 = l_{11} l_{12}, \quad l_3 = -\sin(\psi_1 + \psi_2);$$
$$A_1 = (a''_{13} l_{22} - a''_{12} l_{12}) l_{22}; \quad B_1 = (a''_{11} l_{22} - a''_{13} l_{12}) l_{12}; \quad C_1 = a''_{12} l_{12}^2 - a''_{11} l_{22}^2;$$
$$A_2 = (a''_{12} l_{11} - a''_{13} l_{21}) l_{21}; \quad B_2 = (a''_{13} l_{11} - a''_{11} l_{21}) l_{11}; \quad C_2 = a''_{11} l_{21}^2 - a''_{12} l_{11}^2.$$

Further, using the dependence (3.9) as a uniform expression for ε_1, we represent Eqs. (1.4) in the form

$$B\varepsilon_{3,\alpha} - k_2 \varepsilon_{3,\beta} + (B_{,\alpha} - k_{2,\beta}) \varepsilon_3 = (F - k_1 \varepsilon_2)_{,\beta}; \tag{3.11}$$
$$[B\varepsilon_{2,\alpha} + (1 + k_1) B_{,\alpha} \varepsilon_2 - k_2 B_{,\alpha} \varepsilon_3 - B_{,\alpha} F]_{,\alpha} = 0. \tag{3.12}$$

Thus, the solution of both statements reduces to the solution of Eqs. (3.11), (3.12). If

$$k_2 = 0, \tag{3.13}$$

then, integrating Eqs. (3.11), (3.12), we find

$$\varepsilon_3 = \frac{1}{B} \left[\varphi_3(\beta) + \int_{\alpha_0}^{\alpha} \frac{\partial}{\partial \beta} (F - k_1 \varepsilon_2) \, d\alpha \right]; \tag{3.14}$$

$$\varepsilon_2 = \frac{1}{B} e^{-I} \left\{ \varphi_2(\beta) + \int_{\alpha_0}^{\alpha} \left[\frac{\partial B}{\partial \alpha} F + \varphi_1(\beta) \right] e^I d\alpha \right\}, \quad I = \int_{\alpha_0}^{\alpha} \frac{k_1}{B} \frac{\partial B}{\partial \alpha} d\alpha, \tag{3.15}$$

where $\varphi_1(\beta)$, $\varphi_2(\beta)$, $\varphi_3(\beta)$ are arbitrary functions of the integration.

If $B_{,\alpha} = 0$, then from Eq. (3.12) we find

$$\varepsilon_2 = \alpha \varphi_4(\beta) + \varphi_5(\beta), \tag{3.16}$$

where $\varphi_4(\beta)$, $\varphi_5(\beta)$ are functions of the integration.

In the present case, as the coordinate β we take the lengths of an arc of the β-line. Then, $B = 1$, and Eq. (3.11) can be represented in the form

$$\varepsilon_{3,\alpha} - k_2 \varepsilon_{3,s} = k_{2,s} \varepsilon_3 - f_{,s}, \tag{3.17}$$

where $f = F - k_1 \varepsilon_2$, ε_2 is determined by the dependence (3.16).

Let us examine the possible variants of the Cauchy problem for Eq. (3.17), and their solution:

1) $k_2 = k_2(\alpha)$, $f = f(\alpha)$, $\varepsilon_3|_{\alpha=\alpha_0} = \varepsilon(s)$ (here and below ε is a given function of one variable).

Solution $\varepsilon_3 = \varepsilon\left(s + \int_{\alpha_0}^{\alpha} k_2 d\alpha\right)$;

2) $k_2 = k_2(s)$, $f = f(\alpha)$, $\varepsilon_3|_{s=s_0} = \varepsilon(\alpha)$.

Solution $\varepsilon_3 = \frac{k_2(s_0)}{k_2(s)} \varepsilon\left(\alpha + \int_{s_0}^{s} \frac{ds}{k_2}\right)$;

3) $k_2 = k_2(\alpha)$, $\partial f / \partial s = \xi(\alpha)$, $\varepsilon_3|_{\alpha=\alpha_0} = \varepsilon(s)$.

Solution $\varepsilon_3 = \varepsilon\left(s + \int_{\alpha_0}^{\alpha} k_2 d\alpha\right) - \int_{\alpha_0}^{\alpha} \xi d\alpha$;

4) $k_2 = \text{const}$, $f = f(s)$, $\varepsilon_3|_{\alpha=\alpha_0} = \varepsilon(s)$.

Solution $\varepsilon_3 = k_2^{-1} \{f(s) - f[s + (\alpha - \alpha_0) k_2]\} + \varepsilon[s + (\alpha - \alpha_0) k_2]$;

5) $k_2 = k_2(s)$, $f = f(s)$, $\varepsilon_3|_{s=s_0} = \varepsilon(\alpha)$.

Solution $\varepsilon_3 = k_2^{-1}(s) \left\{ k_2(s_0) \varepsilon\left[\alpha + \int_{s_0}^{s} k_2^{-1}(s) ds\right] + f(s) - f(s_0) \right\}$.

If

$$k=k_2B_{,\alpha}\neq 0, \tag{3.18}$$

then from (3.11), (3.12) we obtain

$$\varepsilon_3 = a_1\varepsilon_{2,\alpha} + a_2\varepsilon_2 + a_3;\ \ b_1\varepsilon_{2,\alpha\alpha} + b_2\varepsilon_{2,\alpha\beta} + b_3\varepsilon_{2,\alpha} + b_4\varepsilon_{2,\beta} + b_5\varepsilon_2 = b_6, \tag{3.19}$$

where $b_1 = a_1B$, $b_2 = -k_2a_1$, $b_3 = Ba_2 + (a_1B)_{,\alpha} - (a_1k_2)_{,\beta}$, $b_4 = k_1 - a_2k_2$, $b_5 = (a_2B)_{,\alpha} - (k_2a_2)_{,\beta} + k_{1,\beta}$, $b_6 = (F + k_2a_3)_{,\beta} - (a_3B)_{,\alpha}$; $a_1 = Bk^{-1}$, $a_2 = (1 + k_1)k^{-1}$, $a_3 = -(B_{,\alpha}F + \varphi_6)k^{-1}$; $\varphi_6(\beta)$ is an arbitrary function of the integration.

Equation (3.19) is solvable for both statements in the case where the inequality (3.18) holds.

In all cases, after the functions ε_2 and ε_3 have been found, using dependences (3.4) or (3.9) we find the function ε_1. Further, for the first statement, we determine the sought functions h and G_1 using formulas (3.3), and, for the second statement, the sought functions G_1 and G_2 using formulas (3.8).

As an illustration of the results obtained for the first statement, let us consider an example. A closed cylindrical shell of elliptical transverse cross section (semiaxes a_1 and a_2, $a_1 > a_2$) and a length L is loaded by a uniform normal pressure ($p_1 = p_2 = 0$, $p_3 = p = \text{const}$, $p > 0$). A characteristic layer of the shell contains only one family of filaments with a rigidity G_1 and an angle of reinforcement ψ. At the end of the cylinder $\alpha = 0$, $\alpha = L$, the following conditions are given:

$$T_1|_{\alpha=0} = T_1|_{\alpha=L} = T_1^* = \text{const},\quad T_3|_{\alpha=0} = -T_3|_{\alpha=L},\quad \varepsilon_2|_{\alpha=0} = \varepsilon_2|_{\alpha=L}, \tag{3.20}$$

and, at the center, the conditions

$$h|_{\alpha=L/2} = h_0 = \text{const},\quad G_1|_{\alpha=L/2} = 0. \tag{3.21}$$

It is required to select the thickness h and the rigidity G_1 in such a way that the cylinder will work in a momentless manner.

Let the angle of reinforcement satisfy the equality

$$\operatorname{ctg}^2\psi = T_1T_2^{-1} \tag{3.22}$$

(here it is obvious that only cylinders and loads can be considered for which $T_1 \geq 0$). Since, in accordance with the condition of the example, there is no preliminary reinforcement ($G_k = 0$, $k = 2, \ldots, N$), from relationships (3.2), (3.5), with the condition (3.22), there follows the equality (3.13).

Then, taking as the parameter β the length of an arc s of the directrix, and using the boundary conditions (3.20) and the conditions (3.21), from (3.3)-(3.5), (1.2), (3.14), (3.15), (1.1) we obtain

$$2h = 2(1+\nu)\Delta\{a\mathrm{E}[2(1+\nu)\varepsilon_2\sin\psi\cos\psi + (\nu\cos^2\psi - \sin^2\psi)\varepsilon_3]\}^{-1}; \tag{3.23}$$

$$G_1 = [2(1+\nu)\varepsilon_2T_3 + (\nu T_1 - T_2)\varepsilon_3](\Delta e_1)^{-1}; \tag{3.24}$$

$$\varepsilon_1 = -k_1\varepsilon_2,\quad \varepsilon_2 = \frac{T_2^0 - \nu T_1^0}{2h_0a\mathrm{E}},\quad \varepsilon_3 = -\int\limits_{L/2}^{\alpha}\frac{\partial}{\partial s}(k_1\varepsilon_2)\,d\alpha; \tag{3.25}$$

$$k_1 = (\nu - \operatorname{ctg}^2\psi)(1 - \nu\operatorname{ctg}^2\psi)^{-1};$$

$$T_1 = \frac{\alpha}{2}(\alpha - L)\frac{d^2R}{ds^2}p + T_1^*,\quad T_2 = Rp,\quad T_3 = \left(\frac{L}{2} - \alpha\right)\frac{dR}{ds}p,$$

where $T_i^0 = T_i|_{\alpha=L/2}$ ($i = 1, 2$).

We introduce the dimensionless quantities

$$H = hh_0^{-1};\quad x = \alpha L^{-1};\quad l = La_1^{-1};\quad t = T_1^*(a_1p)^{-1}; \tag{3.26}$$

$$z_0 = 2h_0a\mathrm{E}[a_1p(1-\nu^2)]^{-1}\left[(\varepsilon_1 + \nu\varepsilon_2)^2 + (\varepsilon_2 + \nu\varepsilon_1)^2 - (\varepsilon_1 + \nu\varepsilon_2)(\varepsilon_2 + \nu\varepsilon_1) + 0.75(1+\nu)^{-2}\varepsilon_3^2\right]^{1/2};\quad z_1 = 2h_0a\mathrm{E}e_1(a_1p)^{-1}$$

We assume that a character layer is reinforced in accordance with the following scheme (Fig. 2):

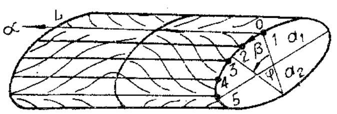

Fig. 2

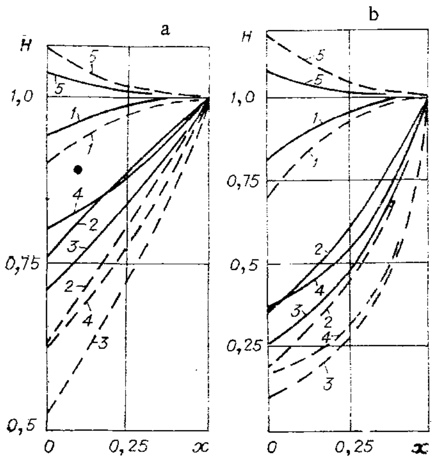

Fig. 3

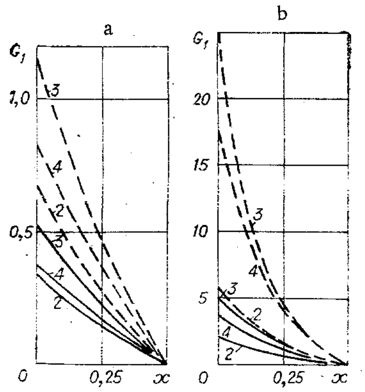

Fig. 4

TABLE 2

l	γ	φ	$x=0$	0,1	0,2	0,3	0,4	0,49
2	0,8	$\pi/8$	0,266	0,264	0,263	0,263	0,262	0,262
		$\pi/4$	0,235	0,236	0,239	0,233	0,236	0,236
		$3/8\pi$	0,212	0,214	0,215	0,216	0,216	0,217
	0,7	$\pi/8$	0,271	0,269	0,268	0,267	0,266	0,266
		$\pi/4$	0,222	0,223	0,224	0,224	0,225	0,225
		$3/8\pi$	0,189	0,192	0,194	0,196	0,197	0,198
4	0,8	$\pi/8$	0,265	0,261	0,257	0,254	0,253	0,253
		$\pi/4$	0,235	0,237	0,238	0,239	0,240	0,240
		$3/8\pi$	0,212	0,219	0,224	0,229	0,232	0,233
	0,7	$\pi/8$	0,271	0,264	0,259	0,256	0,254	0,253
		$\pi/4$	0,222	0,227	0,231	0,237	0,236	0,236
		$3/8\pi$	0,189	0,202	0,213	0,223	0,229	0,231

a) in the sections $0 \leq \varphi \leq \pi/2$ and $\pi < \varphi \leq 3/2\pi$ with $0 \leq x \leq 1/2$ $\psi = -\Omega$, and $1/2 < x \leq 1$ $\psi = \Omega$;

b) in the sections $\pi/2 < \varphi \leq \pi$ and $3/2\pi < \varphi < 2\pi$ with $0 \leq x \leq 1/2 \varphi = \Omega$, with $1/2 < x \leq \varphi = -\Omega$,

where $\Omega = \operatorname{arccot} \sqrt{T_1 T_2^{-1}}$.

The dependence of H on x, calculated using formulas (3.23), (2.17), is shown in Fig. 3a (with $l=2$) and Fig. 3b (with $l=4$). The solid and dashed curves correspond to the cases $\gamma=0.8$; 0.7. The numbers 1-5 correspond to the cross sections $\varphi=0$, $\pi/8$, $\pi/4$, $3/8\pi$, $\pi/2$ (Fig. 2). For determinacy in the calculation, it was assumed that $\nu=0.3$, $t=15$.

The dependence of G_1 on x, calculated using formulas (3.24), (2.17), is shown in Fig. 4a (with $l=2$) and in Fig. 4b (with $l=4$). Here the notation is the same as for Fig. 3. We note that, with $\varphi=0$, $\pi/2$, from (3.25), (2.17) we obtain $G_1 = 0$ with any given values of l, γ, x.

With $\gamma=1$ (round cylinder), from (3.23), (3.24) it follows that $H \equiv 1$, $G_1 = 0$.

Table 2 gives values of the angle Ω as a function of x.

According to [1], the binder and the filaments will work elastically if the following inequalities hold:

$$z_0 < 2h_0 a\sigma_0 (a_1 p)^{-1};\quad 2h_0 aE\sigma_1^- (E_1 a_1 p^{-1}) < z_1 < 2h_0 aE\sigma_1^+ (E_1 a_1 p)^{-1}.$$

Maximal values of z_0 and z_1 along the cross sections $\varphi=\text{const}$, calculated using formulas (3.26), are given in Table 3. The numerator gives values for z_0, and the denominator, for z_1.

It must be noted that, in a given cross section $x=\text{const}$, each of the quantities H, G_1, z_0, z_1 has identical values at points of the middle surface which are symmetrical with respect to the axes of the ellipse of such a cross section. In addition, along the cross sections $\varphi=\text{const}$, the equalities hold $H(x)=H(1-x)$; $G_1(x)=G_1(1-x)$, $z_0(x)=z_0(1-x)$; $z_1(x)=z_1(1-x)$.

As an illustration of the results obtained for the second statement, let us consider an example. A closed cylindrical shell of elliptical transverse cross section (semiaxes a_1, a_2; $a_1 > a_2$) has a constant thick-

TABLE 3

l	γ	φ=0	π/8	π/4	3/8 π	π/2
2	0,8	15,31	15,01 / 13,82	14,43 / 13,58	14,01 / 13,31	13,85
	0,7	15,87	15,25 / 13,92	14,25 / 13,42	13,46 / 12,91	13,20
4	0,8	18,01	17,77 / 15,03	16,79 / 13,08	13,67 / 11,35	11,32
	0,7	20,46	21,05 / 15,60	22,16 / 12,04	16,99 / 9,12	8,52

TABLE 4

l	γ	φ=0	π/8	π/4	3/8 π	π/2
1	0,8	36,37	36,31	36,23	36,23	36,23
	0,75	36,41	36,33	36,23	36,24	36,24
2	0,8	36,82	36,59	36,23	36,23	36,23
	0,75	37,0	36,66	36,24	36,24	36,23

TABLE 5

l	γ	φ=0	π/8	π/4	3/8 π	π/2
1	0,8	—0,481 / —0,481	—0,482 / —0,498	—0,492 / —0,512	—0,499 / —0,511	—0,501 / —0,501
	0,75	—0,479 / —0,479	—0,481 / —0,503	—0,494 / —0,522	—0,502 / —0,519	—0,504 / —0,504
2	0,8	—0,502 / —0,502	—0,486 / —0,614	—0,491 / —0,666	—0,492 / —0,612	—0,493 / —0,493
	0,75	—0,506 / —0,506	—0,486 / —0,659	—0,491 / —0,737	—0,492 / —0,666	—0,492 / —0,492

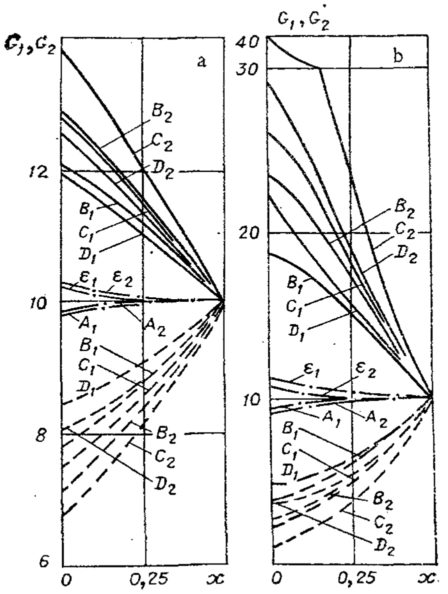

Fig. 5

ness (h = const) and is loaded by a uniform normal pressure ($p_1 = p_2 = 0$, $p_3 = p = \text{const}$, $p > 0$). A characteristic reinforced layer contains two families of filaments with the rigidities G_1 and G_2 and angles of reinforcement ψ_1 and ψ_2; here

$$\psi_1 = -\psi_2 = \psi = \text{const}. \tag{3.27}$$

At the edges of the shell $\alpha = 0$, $\alpha = L$ (L is the length of the cylinder), the following condition with $\alpha = L/2$ is imposed:

$$G_1 = G_2 = G = \text{const}. \tag{3.28}$$

It is required to select the rigidities G_1 and G_2 in such a way that a momentless stressed state will be attained in the cylinder under consideration.

Since, in accordance with the condition of the given example, there is no preliminary reinforcement ($G_k = 0$, $k = 3, \ldots, N$), from (3.7), (3.10), with the condition (3.27), there follows the equality (3.13). Then, as the parameter β, taking the length of an arc of the directrix s, and using the boundary conditions (3.20) and the conditions (3.27), (3.28) from (3.8)–(3.10), (1.2), (3.14), (3.15), we obtain

$$G_1=(\Phi_1+\Phi_2)(\Delta_* e_1 \sin 2\psi)^{-1},\ G_2=(\Phi_1-\Phi_2)(\Delta_* e_2 \sin 2\psi)^{-1}; \tag{3.29}$$
$$\Phi_1=[(2ha\mathrm{E})^{-1}(\nu T_1-T_2)+\varepsilon_2]\sin\psi\cos\psi,$$
$$\Phi_2=[T_3(2ha\mathrm{E})^{-1}-0{,}5(1+\nu)^{-1}\varepsilon_3]\Delta_*;\ \Delta_*=\nu\cos^2\psi-\sin^2\psi;$$
$$e_1=\varepsilon_1\cos^2\psi+\varepsilon_2\sin^2\psi+\varepsilon_3\sin\psi\cos\psi,$$
$$e_2=\varepsilon_1\cos^2\psi+\varepsilon_2\sin^2\psi-\varepsilon_3\sin\psi\cos\psi,$$
$$\varepsilon_1=F-k_1\varepsilon_2,\quad \varepsilon_2=\left(a_{11}^0T_2^0-a_{12}^0T_1^0\right)(2ha\mathrm{E})^{-1}\left[a_{11}^0a_{22}^0-\left(a_{12}^0\right)^2\right]^{-1},$$
$$\varepsilon_3=-\int_{L/2}^{\alpha}\frac{\partial}{\partial s}(F-k_1\varepsilon_2)\,d\alpha;\quad F=\frac{1-\nu^2}{2haE\Delta_*}(T_2\cos^2\psi-T_1\sin^2\psi),$$
$$k_1=(\cos^2\psi-\nu\sin^2\psi)\Delta_*^{-1},\quad T_i^0=T_i|_{\alpha=L/2}\quad(i=1,2);$$
$$a_{11}^0=(1-\nu^2)^{-1}+2G\cos^4\psi,\quad a_{22}^0=(1-\nu^2)^{-1}+2G\sin^4\psi,$$
$$a_{12}^0=\nu(1-\nu^2)^{-1}+2G\sin^2\psi\cos^2\psi.$$

The stresses T_1, T_2, T_3 are determined by formulas (3.25).

We introduce the dimensionless quantities

$$l=La_1^{-1};\quad t=T_1^*(a_1p)^{-1};\quad x=\alpha L^{-1}; \tag{3.30}$$

$$z_0=2ha\mathrm{E}(1-\nu^2)a_1p]^{-1}\left[(\varepsilon_1+\nu\varepsilon_2)^2+(\varepsilon_2+\nu\varepsilon_1)^2-(\varepsilon_1+\nu\varepsilon_2)(\varepsilon_2+\nu\varepsilon_1)+0.75(1+\nu)^{-2}\varepsilon_3^2\right]^{1/2};$$
$$z_1=2ha\mathrm{E}e_1(a_1p)^{-1},\quad z_2=2ha\mathrm{E}e_2(a_1p)^{-1}. \tag{3.31}$$

The dependences of G_1 (solid lines) and G_2 (dashed lines) on x, calculated using the formulas (3.29), (2.17), are shown in Fig. 5a (with $l=1$) and in Fig. 5b (with $l=2$). The dashed-dot curves correspond to the case $G_1=G_2$. The letters A, B, C, D, E denote the cross sections $\varphi=0$, $\pi/8$, $\pi/4$, $3/8\pi$, $\pi/2$. The subscripts 1, 2 correspond to the values $\gamma=0.8$, 0.75. For determinacy, it was assumed in the calculation that $\nu=0.3$, $G=10$, $\psi=80°$, $t=40$.

In the case where the cylinder is round ($\gamma=1$), from (3.29) we obtain $G_1=G_2=G=10$.

According to [1], for elastic work of the elements of a composite material, the inequalities must be satisfied

$$z_0<2ha\sigma_0(a_1p)^{-1};$$
$$2ha\mathrm{E}\sigma_1^{-1}(\mathrm{E}_1a_1p)^{-1}<z_1<2ha\mathrm{E}\sigma_1^{+}(\mathrm{E}_1a_1p)^{-1};$$
$$2ha\mathrm{E}\sigma_2^{-}(\mathrm{E}_2a_1p)^{-1}<z_2<2ha\mathrm{E}\sigma_2^{+}(\mathrm{E}_2a_1p)^{-1}.$$

In the numerical example under consideration, formulas (3.21) give $z_1<0$, $z_2<0$, i.e., the reinforcing filaments in the composition of a composite material work in a compressed state ($e_1<0$, $e_2<0$). Table 4 gives maximal values of z_0, and Table 5 maximal values of z_1 (in the numerator) and z_2 (in the denominator) in corresponding cross sections $\varphi=$const, calculated using formulas (3.30), (3.31), (2.17).

It must be noted that, in any given cross section x=const, each of the functions G_1, G_2, z_0, z_1, z_2 has identical values at points of the middle surface which are symmetrical with respect to the axis of the ellipse of the cross section. In addition, along cross sections $\varphi=$const, the equalities hold $G_1(x)=G_2(1-x)$; $z_0(x)=z_0(1-x)$; $z_1(x)=z_2(1-x)$.

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